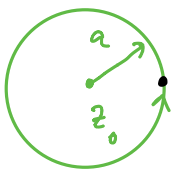
Example 5. Consider a circle of positive radius a centered at any point $z_0 \in \mathbb{C}$. Find an appropriate parameterization which traverses this circle once in the counterclockwise direction and verify one of the most-used contour integral equalities in complex analysis:



$$* \quad \oint_{|z-z_0|=a} \frac{1}{z-z_0}\, dz = 2\pi i.$$

$$\gamma(t) = z_0 + a e^{it}, \quad 0 \le t \le 2\pi$$

$$\frac{\gamma'(t)}{dt} = a i e^{it}\, dt$$

• doesn't matter where we start & end (as long as it's same place)

• limit of Riemann sums.

• or use additivity of integrals over subdomains.

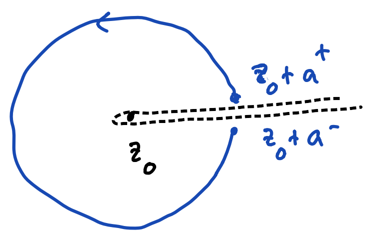
$$* = \int_0^{2\pi} \frac{1}{\gamma(t) - z_0}\, a i e^{it}\, dt$$

$$= \int_0^{2\pi} \frac{1}{\cancel{a e^{it}}}\, \cancel{a} i \cancel{e^{it}}\, dt = i\, 2\pi \quad \blacksquare$$

5b) In an effort to tie this computation in to the FTC for contour integrals, could you compute this integral in that way? (The answer is yes, if you're careful!)

anti deriv of $\frac{1}{z-z_0}$ $\quad \log(z-z_0)$

to use FTC, we'd need a branch domain for $\log(z-z_0)$



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The connection between contour integrals and Calc 3 line integrals:

Let $A \subseteq \mathbb{C}$ open, $f : A \to \mathbb{C}$ continuous, $\gamma : [a, b] \subseteq \mathbb{R} \to \overset{A}{\mathbb{C}}$ a C^1 curve. write

$$\gamma(t) = x(t) + i\,y(t),$$
$$f(z) = u(x, y) + i\,v(x, y).$$

Then

$$\boxed{\int_\gamma f(z)\,\mathrm{d}z} = \boxed{\int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t}$$
$$= \int_a^b \big(u(x(t), y(t)) + i\,v(x(t), y(t))\big)\big(x'(t) + i\,y'(t)\big)\,\mathrm{d}t$$
$$= \int_a^b u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)\,dt$$
$$+ i\int_a^b v(x(t), y(t))x'(t) + u(x(t), y(t))y'(t)\,dt$$

Calc 3

$$\boxed{= \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy.}$$

On Wednesday we'll combine this Calc 3 line integral way of writing contour integrals with Calc 3 Green's Theorem, for some interesting section 2.2 results.

let's go there now

Math 4200
Wednesday September 23

2.1-2.2 Contour integrals and Green's Theorem. We'll finish the examples and discussion related to contour integrals in Monday's notes first, before proceding into today's. As we learned Monday, if an analytic function has an antiderivative, then the value of a contour integral in the domain of that analytic function only depends on the starting and ending points of the contour (FTC for contour integrals). On Friday we'll turn that discussion around to use contour integrals in simply connected domains, to *define* antiderivatives of analytic functions.

Announcements: quiz today
I'll add a Thursday @ 2:00. office hour.

Warm up exercise

For $\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ a C^1 curve and $f : \mathbb{C} \to \mathbb{C}$ continuous, how do you compute

$$\int_\gamma f(z)\ dz\ ?$$

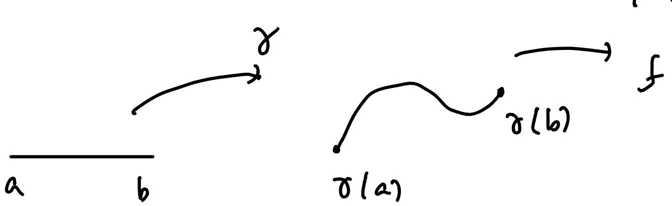
$z = \gamma(t)$
$dz = \gamma'(t)\,dt$

$$= \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

(R.S. interpretation as well, $\lim_{|\Delta z_j| \to 0} \sum_j f(z_j)\,\Delta z_j$)

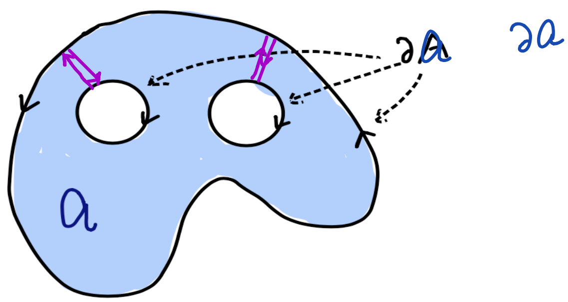
If $f(z) = F'(z)\ \forall\ z$ in our domain, what is the shortcut for computing the contour integral above?

$$= F(\gamma(b)) - F(\gamma(a))$$



Recall *Green's Theorem* for real line integrals from multivariable Calculus, for C^1 vector fields $[P(x,y), Q(x,y)]$ around oriented boundaries of planar domains. A. If you're rusty about why it's true there's a Wikipedia page; I also added an appendix page reminder at the end of today's notes. Depending on the flavor of Math 3220 that you've taken you may also have proven the general Stokes' Theorem about integrals of differential forms, which includes Green's Theorem as a special case. (See Wikipedia *Stokes' Theorem* for a brief overview of this all-inclusive result.)

$$\int_{\delta A} P\,dx + Q\,dy = \iint_A \underbrace{Q_x - P_y}_{} \, dA \qquad \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$



What does Green's Theorem say about $\int_{\delta A} f(z)\, dz$ if f is C^1 and analytic on the closure $\overline{A}$? Hint: Cauchy-Riemann. (Note, we are interpreting the boundary integral as a sum of contour integrals if the boundary has more than one component.)

$f = u + iv$

$$\int_{\partial a} f(z)\,dz \quad \boxed{= \int_\gamma \underbrace{u}_{P}\, dx \underbrace{-\, v}_{Q}\, dy + i \int_\gamma \underbrace{v}_{P}\, dx + \underbrace{u}_{Q}\, dy.} \qquad \iint Q_x - P_y$$

$$\overset{\text{Green's Thm}}{=} \iint_a \underbrace{(-v)_x - u_y}_{\overset{\shortparallel}{0}} \, dA + i \iint_a \underbrace{u_x - v_y}_{\overset{\shortparallel}{0}} \, dA$$

CR2 $v_x = -u_y$ CR1 $u_x = v_y$

$= 0 + i0$

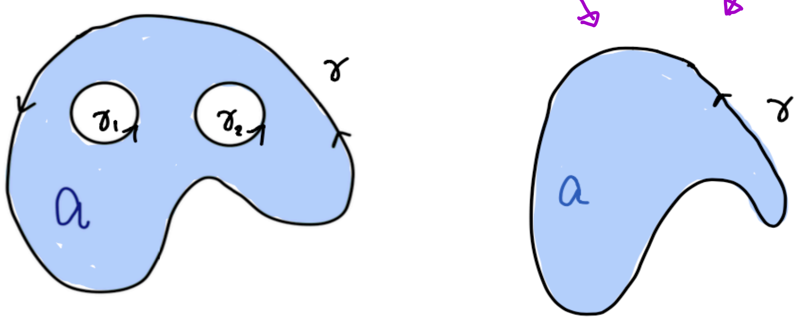
$= 0$!!

Deformation (aka Replacement) Theorem (section 2.2 version): Let $\gamma_1, \gamma_2, \ldots \gamma_n$ be non-overlapping simple closed curves such that γ is a simple closed curve with f analytic in the region between γ and $\gamma_1, \gamma_2, \ldots \gamma_n$ as indicated below. Orient all contours in the counterclockwise definition. Then

$$\int_{\gamma} f(z)\ \mathrm{d}z = \sum_{j=1}^{n} \int_{\gamma_j} f(z)\ \mathrm{d}z$$

If there are no interior curves γ_j the right side is zero and we call this result *Cauchy's Theorem:*

- $$\int_{\gamma} f(z)\ \mathrm{d}z = 0.$$

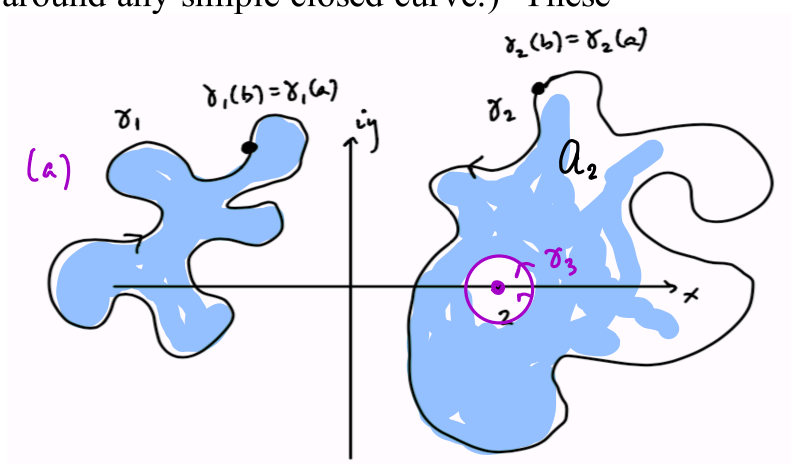


Example 1 Compute

(a) $\displaystyle\int_{\gamma_1} \frac{1}{z-2} dz = 0$ $\quad$ $\frac{1}{z-2}$ is analytic inside γ_1

(b) $\displaystyle\int_{\gamma_2} \frac{1}{z-2} dz$ $\overset{\text{replacement thm}}{=}$ $\displaystyle\int_{\gamma_3} \frac{1}{z-2} dz = 2\pi i$

(You can use partial fractions and these tricks to compute the contour integral of any rational function, around any simple closed curve.) These



Use replacement thm

$a = a_2 \setminus \overline{D(2,r)}$

Contour curve algebra:

Def Let $\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{C}$ be a C^1 curve, with range $\gamma([a, b]) \subseteq A$ an open set. Then we define the curve $-\gamma : [a, b] \to A$ by

$$(-\gamma)(t) = \gamma((a + b - t),$$

$(-\gamma)(a) = \gamma(b)$; $(-\gamma)(b) = \gamma(a)$

i.e. traversing the curve in the opposite direction.

Note, by our discussion of contour integrals in terms of Riemann sums on Monday, if f is a C^1 analytic function on A, then

- $$\int_{-\gamma} f(z)\, dz = -\int_{\gamma} f(z)\, dz .$$

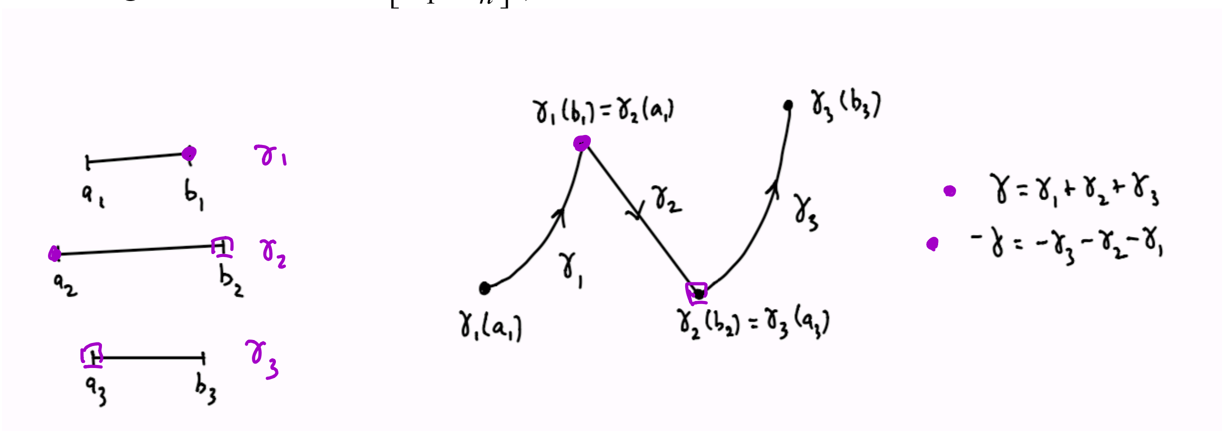
You could also verify this using the definition of contour integrals which converts them into Calculus 1 integrals.

Now, consider piecewise C^1 contours that piece together continuously:

Def: Let $\gamma_j : [a_j, b_j] \to \mathbb{C}$ be C^1, $j = 1, 2 \ldots, n$. Require
$\gamma_j(b_j) = \gamma_{j+1}(a_{j+1}), j = 1, \ldots n-1$

$$\gamma_j(b_j) = \gamma_{j+1}(a_{j+1}), \qquad j = 1, \ldots n-1.$$

Then $\gamma = [\gamma_1, \gamma_2, \ldots \gamma_n]$ will be our notation for the piecewise C^1 path obtained from following the paths in order. (The text acually requires that the intervals $[a_j, b_j]$ piece together as well, i.e. $b_j = a_{j+1}$, which we could always accomplish by reparameterizing the curves if necessary. And in that case γ would *actually* be a piecewise C^1 function on the amalgamated interval $[a_1, b_n]$.)



Def For γ as above, define $\gamma_1(a_1)$ to be the *initial point* of γ, and $\gamma_n(b_n)$ to be the *terminal point.*

Def If $\gamma = [\gamma_1, \gamma_2, \ldots \gamma_n]$ is piecewise C^1 as above we write

$$\gamma := \gamma_1 + \gamma_2 + \ldots + \gamma_n$$

and define the contour $-\gamma$ by $-\gamma = [-\gamma_n, -\gamma_{n-1}, \ldots, -\gamma_2, -\gamma_1]$ so

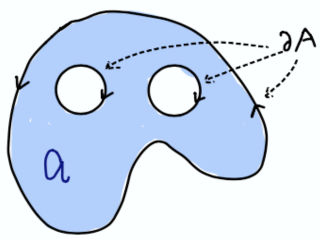
$$-\gamma := -\gamma_n - \gamma_{n-1} - \ldots - \gamma_2 - \gamma_1.$$

And we define the contour integral

$$\int_\gamma f(z)\, dz = \int_{\gamma_1 + \gamma_2 + \ldots + \gamma_n} f(z)\, dz := \sum_{j=1}^{n} \int_{\gamma_j} f(z)\, dz.$$

Greens' Theorem

Let $\langle P(x,y), Q(x,y)\rangle$ be a C^1 vector field, defined on an open domain containing the bounded set $\mathcal{A}$ and its piecewise C^1 boundary $\partial\mathcal{A}$, which we shall write as Γ. Orient (the components of) Γ so that $\mathcal{A}$ is "on the left" as you traverse Γ. (i.e. the pair of vectors given by the tangent to Γ and the inner normal to $\mathcal{A}$ is positively oriented.)



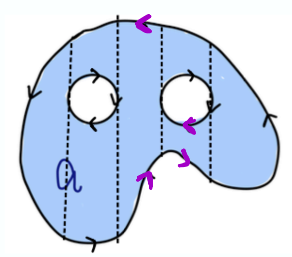
Then

$$\boxed{\oint_{\partial\mathcal{A}} P\,dx + Q\,dy = \iint_{\mathcal{A}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA}$$

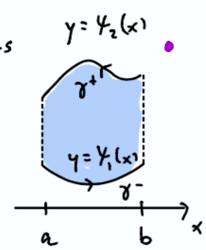
"proof": ① $\displaystyle\oint_{\partial\mathcal{A}} P\,dx = \iint_{\mathcal{A}} -P_y\, dA$ ② $\displaystyle\oint_{\partial\mathcal{A}} Q\,dy = \iint_{\mathcal{A}} Q_x\, dA$

① + ② = Greens' Theorem.

① Partition $\mathcal{A}$ into "vertical simple" subregions



each subregion:



$$\int_a^b \int_{\psi_1(x)}^{\psi_2(x)} -\frac{\partial P}{\partial y}\, dy\, dx$$

$$= \int_a^b -P(x,y)\Big]_{y=\psi_1(x)}^{y=\psi_2(x)} dx$$

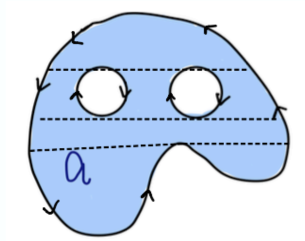
$$= \int_a^b -P(x,\psi_2(x)) + P(x,\psi_1(x))\, dx$$

$$= \int_{\gamma+} P\,dx + \int_{\gamma-} P\,dx$$

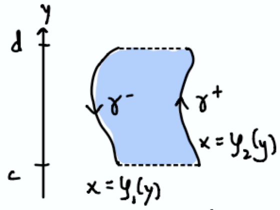
add the identities on the right for each subregion and use the fact that double integrals and line integrals are additive with respect to partitioning.

Deduce $\displaystyle\iint_{\mathcal{A}} -\frac{\partial P}{\partial y}\, dA = \oint_{\partial\mathcal{A}} P\,dx$

② Partition $\mathcal{A}$ into "horizontal simple" subregions



each subregion:



$$\int_c^d \int_{\varphi_1(y)}^{\varphi_2(y)} \frac{\partial Q}{\partial x}\, dx\, dy$$

$$= \int_c^d Q(\varphi_2(y),y) - Q(\varphi_1(y),y)\, dy$$

$$= \int_{\gamma+} Q\,dy - \int_{\gamma-} Q\,dy$$

add identities on the right as in ① to deduce ②